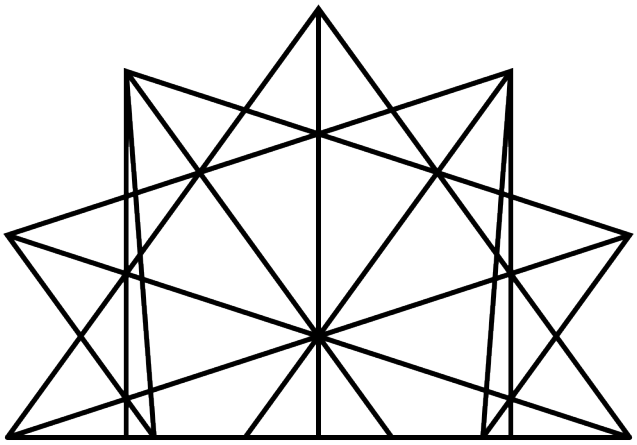
Monochord Procedures



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MONOCHORD PROCEDURES

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INTRODUCTION TO THE MONOCHORD
OVERVIEW OF ITS HISTORICAL IMPORTANCE

The monochord is an ancient instrument whose origin is obscure. It was said by the Greeks to be the invention of Pythagoras; however, like most Greek musical technologies, it was imported, probably from Babylon or Egypt. It is certainly much older than Greek culture, and probably originated in Sumaria.

The monochord was used for a variety of purposes, relating both to musical tuning and scientific-philosophical speculation. It had considerable value as an experimental device. In fact, some historians of science contend that the division of the monochord string was probably one of the earliest empirical-scientific experiments to be carried out with mathematical rigor. As such it was the source of much ancient cosmology and number-lore.

It was also employed as a practical teaching instrument throughout the ancient world and the Middle Ages, in fact right through the Renaissance. The instrument demonstrated the numerical foundation of the consonances and dissonances of music. The musical student also applied it to the study of proper vocal intonation. It had practical applications in musical ensembles. In the West it evolved into the clavichord (hence keyboard instruments in general), the hurdy-gurdy, and the trumpet marine. It was also used as a measuring tool in the design of mediaeval bells and organ pipes.

The monochord's theoretical significance was even more important than its practical influence. It was seen as a symbolic device showing the mystical unity between man (the microcosm) and the universe (the macrocosm). It was the source of much ancient cosmology. The universe was conceived as a stretched string whose pitches represent the solar system (musica mundana), the muses, the zodiac, the parts of the human body, etc. Thus the monochord numbers and divisions were alloted cosmic significance from ancient times right up to the 18th Century. The string was often shown being tuned by the hand of God. The universe was thought to obey musical laws.

Hence we note its practical and theoretical influence on musical culture throughout history. There was no more accurate tool for the measurement of "sonic space" than the stretched string. In fact, more accurate scientific instruments for the calibration of musical intervals did not arrive until the late 19th Century. Now, of course, electronic means of interval measurement are much more accurate than the monochord.

DESCRIPTION

The earliest form of monochord probably consisted of a single string stretched across two fixed bridges on a plank or table (see Figure 1). A movable bridge was then placed underneath the string, dividing it into two vibrating sections. The bridge could divide the sections in any manner, demonstrating that a division by simple fractions resulted in the important consonances. The positions of the movable bridge were calibrated by marks on the table beneath the string. Hence the monochord can best be described as a "musical yardstick" for the measurement of ratios.

From early times the monochord became a multi-stringed instrument tuned in unisons so that comparisons could be made between various divisions. The ancient name KANON is reflected in the modern Persian multi-stringed zither which probably evolved directly from the ancient monochord. The instrument was also known as the "Philosopher's Lute" in classical China. In Japan, the modern KOTO is based on the same movable bridge principle. Thus the monochord was not confined to European culture. Indeed, it is ubiquitous in the ancient world.

In mediaeval Europe, after the 12th Century, the monochord was fitted with a resonating box and made more portable (see Figure 2). After about 1500, monochords were built in which one end-bridge was replaced by a nut so that the string was lowered, enabling the user to press it directly against the belly of the instrument, like a modern guitar. Although it was thus simpler to use, it was considerably less accurate than the traditional monochord. This "popular" form of monochord, however, reflected the resurrgence of interest in the instrument during the renaissance, which was a historical period of much tuning experimentation.

Most mediaeval monochords were from 90 to 120 cm. in length--a length convenient for calibrating even micro-intervals. For example, at 100 cm. string length, the position of a comma as the "first fret" is about 1.2 cm. from the bridge. Thus considerable subtlety was possible, even down to parts of a comma.

The accuracy of the monochord depends upon the skill of the user as well as a number of other factors. For example, the greater the number of steps involved in producing a desired ratio, the less accurate is the result, since errors are accumulative. There is also a built in innacuracy due to the necessity of deflecting the string with the movable bridge. Another innacuracy is inherent in the materials of the string itself. But, in spite of these shortcomings, the instrument is still accurate enough to generate all of the tuning lore of historical tunings and temperaments. This tuning lore had considerable bearing on ancient science and philosophy. It also must have profoundly influenced the development of mathematics and abstract thought.

CALIBRATION METHODS AND CONCEPTUAL MODELS

The divisions of the monochord string can be represented by four methods: proportional ratios, string lengths, logarithms, and cents. The modern division by cents is derived from the 18th Century application of logarithms to music, and cannot be directly applied onto the instrument without further calculation. Hence I will only consider ratios and string lengths here. The calibration by string lengths is derivative from the proportional system. It overcomes the cumbersome nature of fractional ratios by clearing fractions through a least common multiple chosen as the length of the open string.

It is clear that the approach using ratios and the approach using string lengths are equally ancient as well as interchangeable. However, there is a certain numerical complexity inherent in the system of least common multiples, since many different numbers may be used to express the same proportional division. For example, the musical Fifth which is defined by the proportional ratio 2/3 may be represented as 360:540 or as 120:180. In other words, if the open string is 540 units long, the placement of the movable bridge for the Fifth is found at 360 units from the fixed bridge. Again, if the open string is 180 units long, the Fifth is found at 120 units. Both sets of numbers express the simple ratio 2/3 with the fractions cleared. If we wish to express another ratio by an appropriate string length, say the comma ratio 80/81, then we would need to use a different overall string length in order to achieve a least common multiple to "clear" the fraction. Hence different sets of numbers can have the same meaning when we use the method of string lengths by clearing fractions. We are being asked to constantly change our perspective with regard to the meaning of the numbers according to the tuning situation at hand. This approach was used by the Greeks and the Babylonians, even though the latter had no problems handling complex fractions. (They used base 60 for their mathematics, a number base peculiarly suited for monochord investigations).

Obviously, the constant within this flux of musical arithmetic is the ratio itself, expressed in simplest numerical terms. The musical Fifth is found at 2/3rds of the string length, no matter what numerical "index" we use in order to express it. Thus I would consider the ratio itself as of primary interest. For the sake of clarity, I will concentrate on the ratios in this short article, and leave the detailed approach to least-common-multiple string lengths for the companion article entitled MONOCHORD NUMBERS. However, the two articles are meant to be considered together. Each throws light on the other. There will be occasional cross-overs in both articles to clarify matters.

When expressing the ratios we must distinguish between the frequency ratios and the ratios of string length. Frequency is inversely proportional to string length when other factors (eg. string tension) are kept constant. This scientific law is at the heart of the monochord's utility as a measuring device. For an

example of this law, consider that the raising of pitch by a musical Fifth is expressed by the frequency ratio 3/2, but it is found on the monochord by dividing the string by 2/3. Also, the lowering of pitch by a Fifth is expressed by frequency ratio 2/3, but it is found on the monochord in dividing the string by 3/2. Now this division would mean lengthening the string by a third, an impossibility if we are starting with the open string. However, such a "descending" division can be possible if we start with part of the string (for instance, the octave). I will explain these proceedures latter. The main point to grasp here is the fact that we can generate the same information whether we use frequency ratios (as is the modern tendency), or whether we use ratios of string length (as is the ancient approach). Both ways of looking at it show the inherent polarity of musical pitch (up and down). Apparently, the ancients were not aware of frequency, but this was no handicap in their analysis of consonance.

We shall calibrate a monochord string whose active, vibrating length is unity or one. We define the right-hand end of the string as the fixed bridge, and the left-hand end of the string as the "nut," as on a modern guitar. The string, vibrating as a whole, is shown on Figure 3. This diagram illustrates the division of the string by one, resulting in the musical Unison. The string division is shown under the string, and the frequency ratio and musical interval above it. To the right, the resulting musical tones of the two vibrating portions of the string are marked on the Grand Staff. The open string is tuned to our reference pitch of Tenor C.

When the movable bridge is placed in the exact middle of the string, indicating a division into two halves, the musical Octave is produced. See Figure 4. Both the left-hand side and the right-hand side of the divided string sound the Octave above the open string, as shown on the Staff to the right. By this simple division we produce the strongest consonance after the Unison.

Figure 5 and Figure 6 show the two positions which result from the division of the string into three segments. These positions show the Just Musical Fifth and Twelfth which are the next strongest consonances. The two segments of the string relate by an Octave. These placements co-incide with the positions of the Third Harmonic (frequency ratio 3/1) of the string. Note that Figure 6 can be considered redundant since there is no new information not already seen in Figure 5. The notes have simply exchanged places.

A division of the monochord string by four results in three possible placements of the movable bridge. These placements are one-quarter of the distance from the nut, one-quarter of the distance from the bridge, and the mid-point. But the mid-point position was already considered on Figure 4. The other two positions are shown as Figure 7.

The 3/4 position is associated with the Just Musical Fourth. The 1/4 position is the Fourth Harmonic or Double Octave.

Four, of course, is two times two, and there is every reason to believe that the ancients knew that powers of two connect with octaves. Two is the Pythagorean "matrix" number, the "mother" whose cyclical identity is the natal womb of the other musical intervals. Even numbers, divisible by two, were considered female. They do not produce new tones, only octaves of tones already generated by the male, odd numbers. The prime male numbers three and five generate the field of Just Intonation.

The ancient Greek scale system divided musical space into a double octave, the ratio 4/1. Also, the musical Fourth, frequency ratio 4/3, was the all-important tetrachord interval of Greek and Hindu music theory. The octave could be shown to consist of two falling tetrachords, C-G and F-C, connected in disjunct fashion by a 9/8 wholestep between G and F. It was also seen as two falling tetrachords, F-C and C-G, joined in conjunct fashion with an added wholestep to complete the octave. Figure 8 shows the metrical properties of these three pitches, C, F, and G, on the Circular Graph (a logarithmic analog in which intervals expressed as ratios or cents are converted to degrees of arc. 360 degrees is defined as an octave, and clockwise movement indicates falling pitch).

In ancient Greek and Hindu musical theory, the Tetrachords between C, F, and G were considered the axis of fixed tones for the creation of modal scales. The other pitches which divide the Tetrachord into further scalar tones were considered mutable or expressible through alternative ratios. The importance of this central axis is perpetuated in the modern concept of the Tonic, Dominant, and Sub-dominant.

The Pythagorean approach to division by proportions was based on the relationships between the three Means--Arithmetic, Harmonic, and Geometric. The Arithmetic and Harmonic Means result in simple numerical ratios of musical significance in Just Intonation. The Geometric Mean, on the other hand, produces irrational ratios which are associated musically with temperament (controlled mis-tuning of the normative Just ratios). The Arithmetic and Harmonic Means of the Octave give us the Fourth and Fifth. The Geometric Mean gives $\sqrt{2}$--a tempered form of the Just Tritones 7/5, 10/7, 45/32, and 64/45.

The concept of the Mean had great philosophical appeal to the Greeks, Babylonians, and Chinese, who saw life as a dynamic balance between the play of opposites. A finite, orderly mean between two extremes would share in the quality of the extremes, yet bring order out of chaos. The Mean itself would be a direct descendent of the original complimentarity and share in its life energy. It must have been a source of great intellectual pleasure to the Greeks that the Means of the all-important Octave are the important consonances of the musical Fourth and Fifth.

Moreover, the inherent ratio numbers involved, 1, 2, 3, and 4, add up to the Pythagorean sacred number 10 and associate with the revered Tetractys. We shall see that the Tetractys is

also connected with the proportional series 1:2:3:4:5:6:7:8:9:10 which are the consonant ratios of over-riding importance in tuning systems. The implications of the doctrine of Means are explored in detail in the associated article THE MEANS AND MUSIC.

If we wish to demonstrate the important ratios 2/1, 3/2, and 4/3 on the monochord without fractions, we must divide the string into twelve sections. In other words, we must use twelve units as our least common multiple string length. See Figure 9. Here we have the source of the so-called "Musical Proportion" 6:8:9:12, which Pythagoras reputedly brought home from Babylon. These numbers are multiples of the two primes, two and three. Remember that these numbers can apply to both rising and falling sequences of pitch, that is, to both multiples and sub-multiples of the unit string length. Thus F and G can exchange places as Arithmetic and Harmonic Means to C. See Figure 10. These musical inversions function in reciprocal ways as means. From the Circular Graph (Figure 8) note that, in relation to F and G, the generator tone C functions visually as a Geometric Mean. We shall see that C forms such a constant between any interval and its musical inversion. The dotted line on the Circular Graph which connects C and $\sqrt{2}$ is the axis of symmetry for these inversion pairs (more of which are shown on Figure 19).

We can also express the harmonic relations between the ratios 1:2:3:4 through the Field Diagram which is a practical guide to Just Intonation tuning. See Figure 11. The generator pitch C is 1 (the First Harmonic). The pitch G, a pure musical Fifth above C, is found by forming a beatless unison between the Third Harmonic of the C string and the Second Harmonic of the G string, as shown on the Staff. The pitch G, a pure musical Fourth below C, shown in brackets, is found by forming a beatless unison between the Third Harmonic of the C string and the Fourth Harmonic of the G string, as shown on the Staff. Looking at the reciprocals, F a pure Fourth up in pitch is found by forming a beatless unison between the Fourth Harmonic of the C string and the Third Harmonic of the F string. F a musical Fifth down in pitch from C is found by tuning a beatless unison between the Second Harmonic of the C string and the Third Harmonic of the F string. Performing these tuning operations is easier than describing them! These operations may be performed on any stringed instrument, including the monochord.

Lastly, the harmonic relations between ratios 1:2:3:4 may be represented in an even more abstract manner as Figure 12. The central Octave is represented by the double 2-4 in the square brackets, and the reciprocal relations of the number 3 are presented on either side--as an "over number"to the right, ie.3/2, and an "under number" to the left, ie. 2/3. It can easily be seen how this abstract representation of the number field leads to the tuning pattern shown below it. Expansions of similar patterns with a larger double as "index" in the centre, integrate greater portions of the number field and form the Field Diagram of Just Intonation. These patterns were drawn in the sand using pebbles as

place markers and constituting the Pythagorean PSEPHOI-ARITHMETIC (literally "pebble-arithmetic").

These abstract number games of the ancient philosophers were probably all derived from tuning operations on the monochord. This instrument, combined with the Circular Graph and the "pebble" arithmetic, formed a powerful set of interlocking conceptual tools for the description of a musical interpretation of the number field.

The division of the monochord by the prime number five introduces a whole new set of consonances to music. Five is male, like three, but five is characterized as the "human" male number, as opposed to three, which is the "divine" male number. Ratios involving the number five introduce the so-called "Imperfect Consonances" of the Just Major Third (frequency ratio 5/4) and the Just Major Sixth (frequency ratio 5/3). These ratios and all subsequent ones divide the tetrachord into smaller portions.

The four new positions of the movable bridge are shown on Figure 13. The frequency ratio 5/1 is the Fifth Harmonic which is two octaves and a Major Third above the fundamental. 5/2 is an octave and Major Third above, and 5/4 is the Just Major Third itself. 5/3 is the Just Major Sixth above C. In order to produce the inversions of these intervals on the monochord, we must divide the string by six and eight, since the inversion of the 5/3 Major Sixth is the 6/5 Just Minor Third, and the inversion of the 5/4 Major Third is the 8/5 Just Minor Sixth. The relevant monochord positions are shown as Figure 14.

It is clear that tuning operations involving the Fifth Harmonic need a separate tuning axis on the Field Diagram, since powers of three and powers of five are incommensurate. The layout of these important consonances is shown as Figure 15. This pattern is called the Harmonic Heptad, and forms the nucleus of the Field of Just Intonation. (For an extensive description of the Field of Just Intonation and deviations from it, see the associated article THE MEANTONE SERIES OF CYCLICAL TEMPERAMENTS). Just Intonation is defined by ratios formed from numbers which are powers of the numbers two, three, and five. Since powers of two are hidden, due to the principle of Octave Equivalence, the Field of Just Intonation can be modeled after the lattice of the 3 * 5 Multiplication Table. These ratios may all be derived from the correct arithmetical calculations coupled with the appropriate monochord proceedures. I will outline these proceedures after a consideration of the Septimal ratio consonances.

The division of the monochord by seven introduces a whole new set of consonances which are incommensurate with powers of two, three, and five. However, as shown in the article mentioned above, they form "near misses" to certain 3 * 5 Just Ratios. Therefore they are present within the Just Lattice in a "hidden" manner. The number seven was associated with mystery and the hidden in ancient number symbolism, a symbolism justified by its musical character.

The six positions of the movable bridge which result from the division by seven, as well as two positions from the eight division which form septimal ratios, are shown on Figure 16. The ratio 7/1 is the Seventh Harmonic which is a sharp sixth (or sub-minor seventh) above the double octave. Ratios 7/2 and 7/4 represent lower octaves of this interval, 7/4 being the sub-minor Seventh itself. The ratio 7/6 represents the sharpened wholestep or sub-minor Third. 7/3 is its octave transposition, the sub-minor Tenth. The ratio 7/5 is the Just Septimal Tritone or the sub-minor Fifth.

The frequency ratio 8/1 is, of course, the triple octave, being two to the third power. The ratio 8/7 is the Septimal Wholetone or Super-second, the inversion of the Sub-minor Seventh (7/4). Other inversions of Septimal ratios need further divisions of the monochord. For example, the inversion of the 7/5 tritone is the 10/7 Septimal Diminished Fifth. The inversion of the 7/6 Septimal Minor Third is the 12/7 Septimal Major Sixth (the Super-major Sixth).

One more inversion pair of Septimal intervals should be noted. The 9/7 Septimal Major Third (Super-major Third) and its inversion, the 14/9 Septimal Minor Sixth (Sub-minor Sixth or Sharp Fifth) require a different division of the monochord string. These two ratios complete the set of prominant Septimal consonances.

The various positions of the division by nine, without redundancies, are shown on Figure 17. The most important new consonances are the frequency ratios 9/8, the Major Wholetone or Pythagorean Wholetone, and the 9/5 Just Minor Seventh.

The prominant positions of the division by ten are shown on Figure 18. An important new ratio appears which is the inversion of the 9/5 Minor Seventh, namely the 10/9 Just Wholestep or Minor Wholetone. This interval is a syntonic comma flat of the 9/8 Major Wholetone.

The division by ten for the first time brings into relief the comma which is the fundamental building block of Just Intonation and its close cousin--53-Equal-Temperament. Just Intonation is best seen in a practical sense as a scale of commas. Hence the number ten and the ratios 8:9:10 have great importance in tuning theory. Remember that the Tetractys can be interpreted on one level as the set of consonant epimoric ratios contained within the series 1:2:3:4:5:6:7:8:9:10. To the Pythagoreans, the Tetractys (encompassing the first ten numbers) underlied "the pattern and type of all nature" (from Plato's EPINOMIS). The Pythagoreans saw nature in musical terms. Indeed, "musical nature," in the form of the Field Lattice of Just Intonation, is generated through the comma scale. All prominaint forms of musical tempering are best visualized with reference to this lattice. The use of the comma provides an adequate "mapping" of sonic space, as is shown in the article on the MEANTONE SERIES OF CYCLICAL TEMPERAMENTS. Hence the importance of the number ten.

Some of the prominant ratios of Just Intonation are illustrated on the Circular Graph on Figure 19. The Tetrachord frame is drawn in, and inversion pairs of intervals are connected by dotted lines. Accompanying this Graph is the associated Lattice showing the placement of these musical ratios for tuning purposes.

The first ten numbers had special prominance for Pythagoras, Archytas, and Plato. But, of course, we need not stop there in our divisions of the monochord. The next division involves the prime number eleven, and we rightly suspect that a whole new series of musical intervals are introduced. This series of ratios has not been intregated into classical European music culture. On the other hand, these "neutral" intervals have been used in Persian and Middle Eastern music, as well as some classical Greek scales.

The ten positions of the movable bridge associated with this division are shown on Figure 20. Figure 21 shows the prominant intervals associated with the Eleventh Harmonic, as well as their placement on the Circular Graph. We associate these ratios with the "neutral" intervals which fall approximately between the major and minor forms.

With these ratios we are approaching a saturation of the Harmonic Field. For example, ratios 11/10 and 12/11 are only 14.4 cents different in pitch. Also, 11/7 and the Just Sharp Fifth (ratio 25/16) are only 9.9 cents different in pitch. All of these ratios derived from the Eleventh Harmonic are sonically close to some pitch from the extended Just Field. It seems that we have stepped through a "harmonic boundary" in going beyond ten. It could be argued that we have already witnessed saturation with the Septimal Ratios, since they closely approximated extended Just ratios. However, the Septimal Ratios can be integrated into the Just Field of commas with only slight innacuracy; whereas, the Eleventh Harmonic ratios (and beyond!) cannot be integrated with the same ease. The Septimal Ratios are also much more important from a practical musical standpoint. The neutral intervals are rightly associated with the Harmonic Antipodes of the Just Field. They represent the "outer reaches" of the field of harmony.

Ratios based on higher prime numbers, eg. 13, 17, 19, and so on, quickly saturate the Harmonic Field. The complexity becomes unmanageable. Hence, we will not continue with further prime numbered divisions of the monochord. We are again reminded of the Pythagorean association between the number ten and the macrocosm.

MONOCHORD PROCEEDURES

Any of the ratios of the Just Field, no matter how complex, can be found on the monochord if one uses the proper proceedure aided with some elementary arithmetic. I will give some examples, both simple and complex.

Suppose we wish to find the sum of the Major Wholetone (frequency ratio 9/8) and the Minor Wholetone (10/9). To add musical intervals we multiply their ratios. Thus 9/8 * 10/9 = 5/4, the Just Major Third. In order to apply this computation to the monochord, invert the ratios so that we deal with string lengths. Now 8/9 * 9/10 = 4/5. We now have the choice of measuring 4/5 directly on the instrument or of applying the multiplication on the instrument. This second approach is less accurate because we must take two measurements rather than one. Errors are accumulative. The proceedure is shown on Figure 22.

In this case it is definitely easier to measure the 4/5 directly on the instrument instead of going through the multiplication proceedure. On the other hand, if we were adding 9/8 and 9/8 to get the dissonant Pythagorean Major Third (81/64), then the two step method may be preferable.

We can also subtract intervals on the monochord. Suppose we wish to subtract a Sub-minor Third from the Octave. In frequency ratios the computation is 2/1 * 6/7 = 12/7 (the Septimal Major Sixth). The monochord proceedure is shown on Figure 23. In this case it probably would have been easier to do the arithmetic and then measure the 7/12 position directly. However, given a more complex computation--for instance, the Septimal Sixth subtract the Just Major Third (frequency ratios 12/7 * 4/5) to get the dissonance 48/35--a direct measurement is not as easy as the two step process.

We can only subtract intervals if the remainder is above the open string in pitch, since the string cannot be lengthened. Thus a computation like a Just Fourth subtract a Just Fifth (frequency ratios 4/3 * 2/3 = 8/9) is impossible since we cannot move the bridge to 9/8ths of the open string length. Aside from this limitation, however, any subtraction is possible.

The ancient writers associated the proceedures which result in falling pitch with a "speculative" use of the monochord, and the proceedures associated with rising pitch with the "practical" use of the instrument. This dichotomy must have been yet another expression of the inherent polarity of musical pitch. Obviously, computations which involved both lengthening and shortening the string were accomplished. The Greeks, however, did theorize the scale as a falling pattern, so that its application on the monochord involved starting with the Octave (string ratio 1/2) and progressively lengthening the string to reach the next lower Octave (the open string). The Mediaeval theorists reversed the process. We inherited the Mediaeval view, so that we see the scale as a rising progression.

The Greeks thus started their tunings from the middle of the string and worked down. But we have ample evidence that the given tuning also went "up" as well as down. The Greeks saw the "musical world" as a two octave structure. We are reminded of the quote from Philolaos: "The world is One; it began to develop from

the middle." In other words, tuning theory begins with Unity, but on the practical level it starts at the Octave and proceeds down.

Whole strings of computations are possible on the monochord if the user is comfortable with the arithmetic involved. For example, suppose we wish to "sound" the Pythagorean Semitone or Limma (frequency ratio 256/243). Now here is a complex ratio to measure directly, but its derivation is simple enough. It is the interval "left over" when two Pythagorean Wholetones (frequency ratios 9/8 * 9/8 = 81/64) are subtracted from a pure Fourth (4/3). Thus the problem is best pictured in frequency ratios as 4/3 * 8/9 * 8/9, or in string lengths as 3/4 * 9/8 * 9/8--a simple enough manouvre when done on the instrument. See Figure 24.

It seems quite likely that the ancient tuner "roamed" over the entire Harmonic Field of Just Intonation and was aware of all the important boundaries involved, namely the Schisma interval, and the Enharmonic Gateway. The morphological structure of this Field was expressed in alleqory through the old writings of Greece and the Near East. The monochord and the pebble arithmetic went hand in hand to "map" sonic space through numbers. These musical numbers were then extrapolated into the fields of cosmology, astronomy, engineering, architecture, politics, mythology, and so on.

THE LAMBDOMA

I conclude this introduction to the monochord with a brief consideration of the ancient PYTHAGOREAN TABLE or LAMBDOMA. The Neo-Platonist writer Iamblichus named it the Lambdoma because of its similarity in shape to the Greek letter Lambda. It is a clear and concise way of expressing the monochord ratios in a manner which balances the polarity between string division and multiplication (or the polarity between the Harmonic Series and the Sub-harmonic Series). This table was rediscovered in the 19th Century by Albert von Thimus (DIE HARMONIKALE SYMBOLIK DES ALTHERTHUMS, 1868-72). Its true antiquity is unknown.

This table, along with the Field Diagram, is an important key to Pythagorean Philosophy. For even though the monochord string could only be divided in actual practice, it could be multiplied through inductive reasoning once polarity is established. The table shows the symmetry between the two operations.

The Lambdoma is presented as a ratio field on Figure 25, and as its equivalent field of pitches on Figure 26. I have used a limiting index of twelve, but the table could be extended theoretically to infinity. Lines are drawn connecting some identical tones. For example, 1/1, 2/2, 3/3, 4/4 and so, or again, 1/2, 2/4, 3/6, 4/8, 5/10 and so on. These lines are called identity rays. Note that they all converge at one point which could be considered the source, or 0/0--silence. The manifest world has its origin in the One (Monad) which becomes two (the

Octave). A primary trinity exists between 1/1, 1/2, and 2/1. The numbers one and two were considered "principles" of number rather than numbers themselves. One is the Divinity from which all manifest nature comes. Two is the matrix which creates the essential polarity of existence. Three is the first number, and it is sufficient to produce the "multitude," for operations involving powers of three divide the octave into (eventually) the comma scale. Five is the human number because the Schisma interval can be considered subliminal to human hearing, hence the integration of the 3 * 5 Harmonic Field. Note that the natural Major Triad stems from the Third, Fourth, and Fifth Harmonics. The Just Major and Minor Triad is the archetype and paradigm of the Harmonic Field.

The Lambdoma has a direct connection with the monochord. It is less abstract than the Field Diagram, because the Field Diagram uses Octave Equivalence to simplify the expression of the 3 * 5 number field. In the Lambdoma we are again reminded of the pattern of the Tetractys:

```
                1/1
            2/1     1/2
        3/1     2/2     1/3
    4/1     3/2     2/3     1/4
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It is clear that the Tetractys, as a symbol, can be interpreted on many levels, for it also appears as an important pattern in the Field Diagram.

The Lambdoma logically summerizes the operations on the monochord when we conceive the ratios as proper fractions of string length. However, if we use a least common multiple to clear fractions on the instrument, then the Field Diagram naturally emerges as the most logical "map" of the Harmonic Field. Hence the Field Diagram takes precedence in the article on MONOCHORD NUMBERS.

BIBLIOGRAPHY OF LITERATURE CITED

Crocker, Richard: PYTHAGOREAN MATHEMATICS AND MUSIC; Journal of Aesthetics and Art Criticism, XXll (1963).

Heath, Sir Thomas: A HISTORY OF GREEK MATHEMATICS; Vol. 1, Oxford Clarendon Press, 1921.

McClain, Ernst G: THE PYTHAGOREAN PLATO; Nicolas Hayes, N.Y. 1978.

Plato: THE EPINOMIS; Thomas Taylor, N.Y. Pantheon, 1944.

Figure 1:

A primitive monochord:

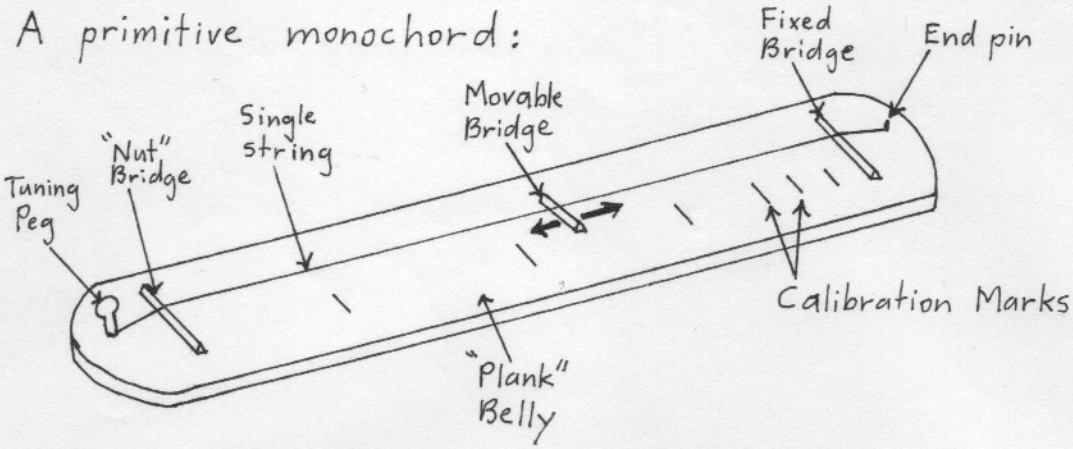


Figure 2:

A more complex, mediaeval-style monochord:

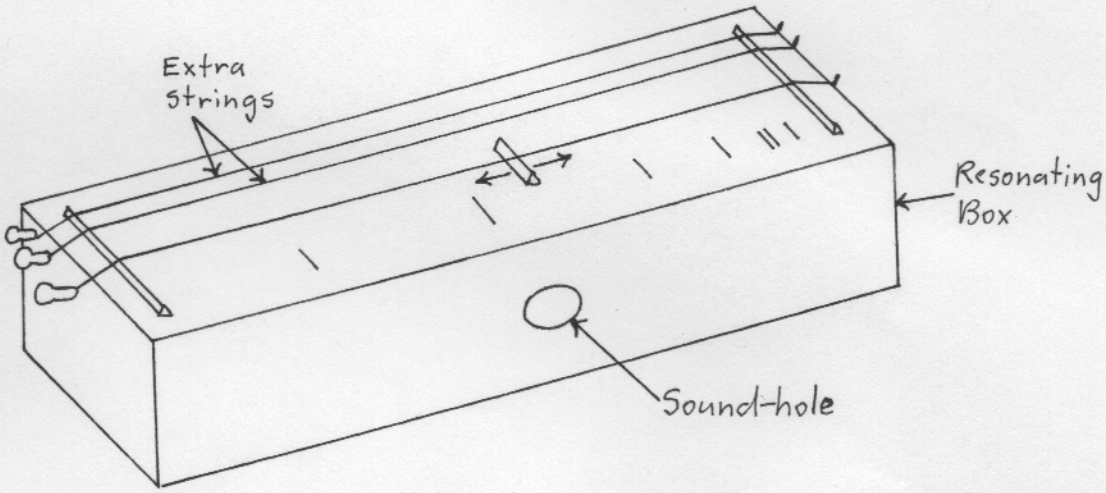
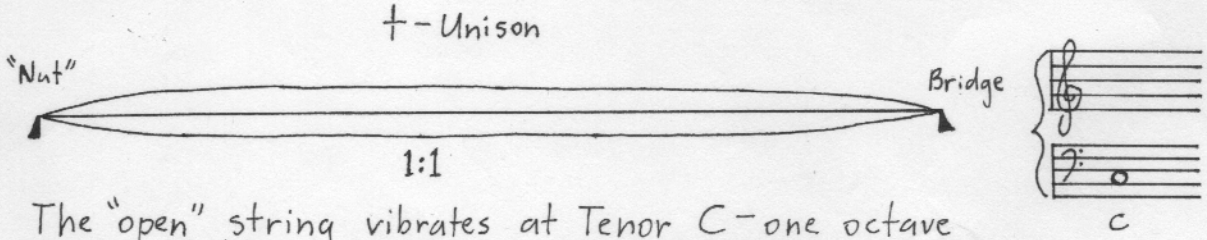


Figure 3:



The "open" string vibrates at Tenor C - one octave below Middle C. The "division" by One is associated with the Monad.

Figure 4:

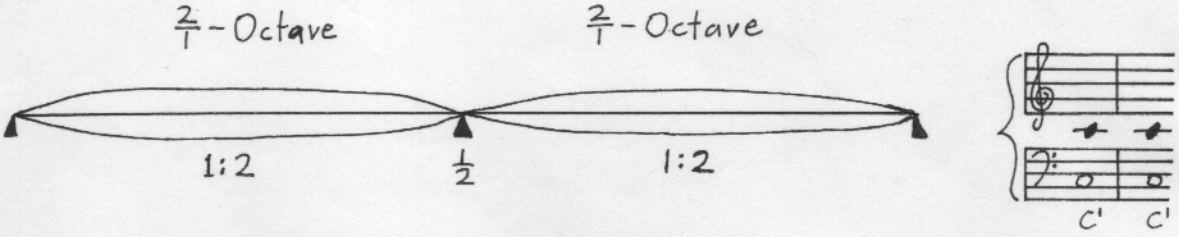


Figure 5:

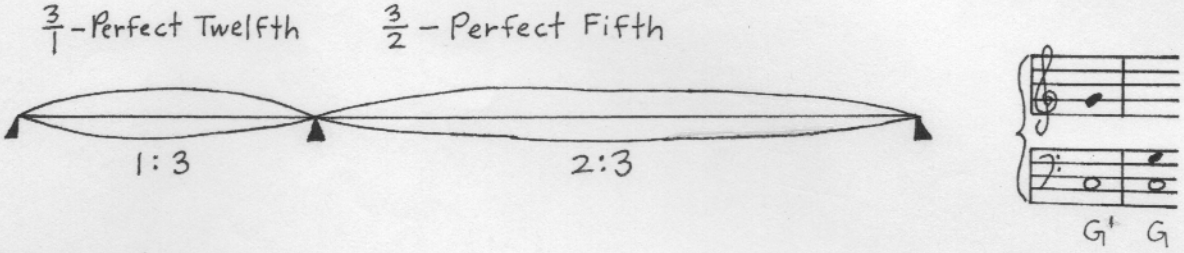


Figure 6:

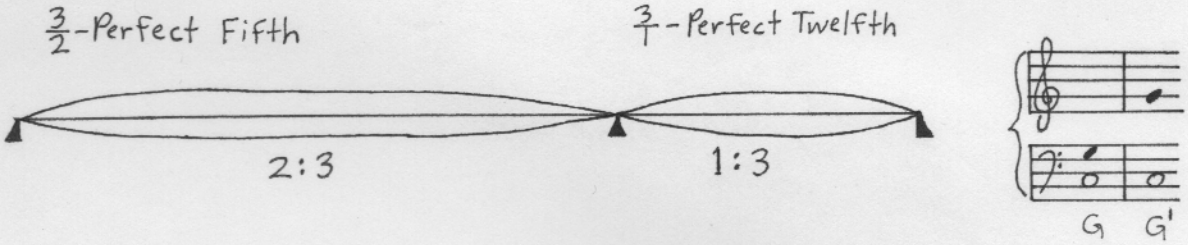
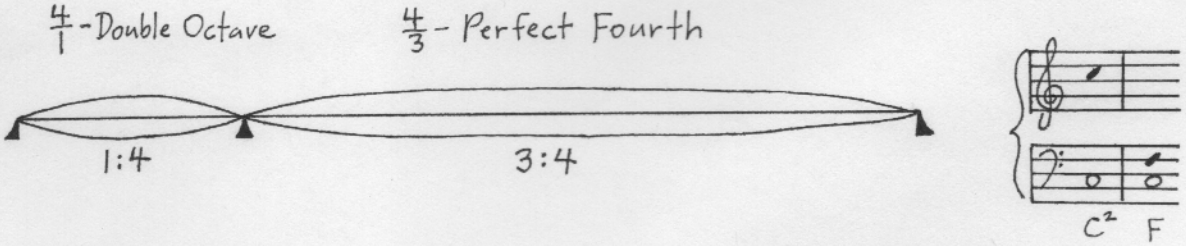


Figure 7:



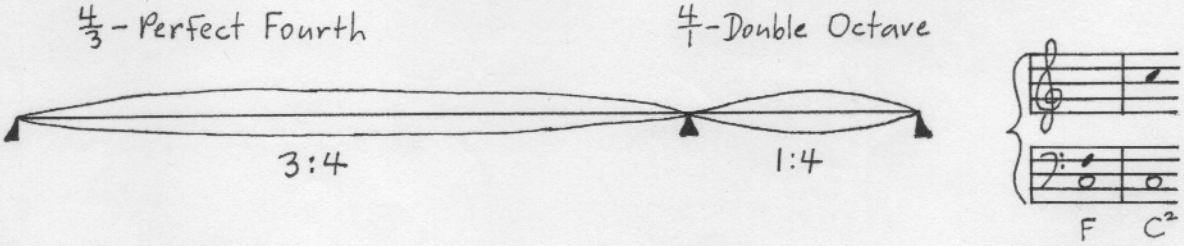


Figure 14:

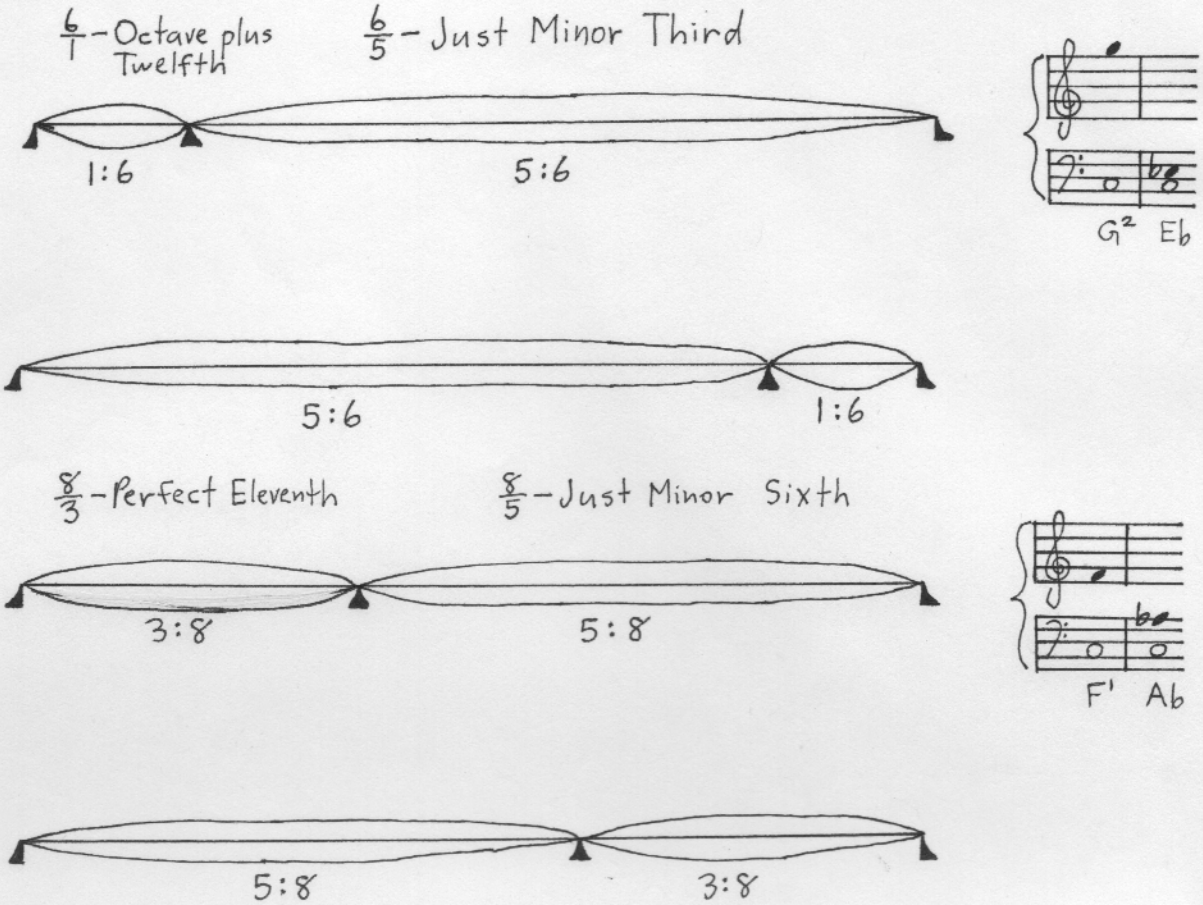
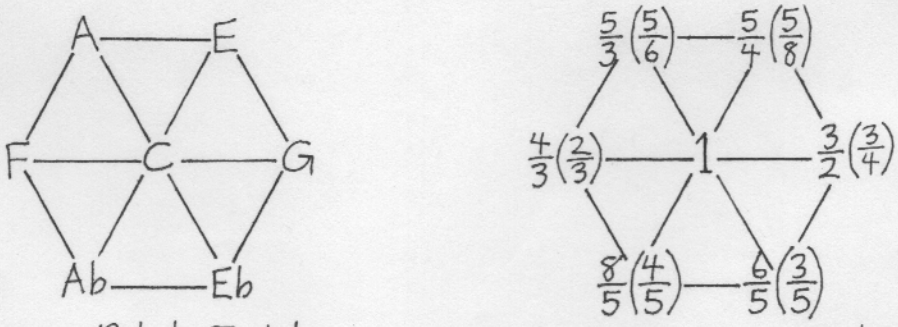


Figure 15: The Harmonic Heptad – the nucleus of the field of Just Intonation:



As a Pitch Field

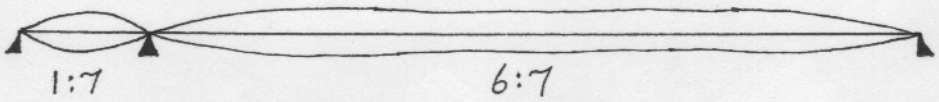
As a Field of Ratios for Just Tuning by Harmonics

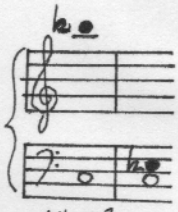
These tuning operations may be extended in all six directions indefinitely (since the field of $2^x \cdot 3^y \cdot 5^z$ numbers extends toward infinity; however, definite boundaries are encountered which imply the various systems of Temperament (controlled mis-tuning of the Just Field).

Figure 16: Division of the monochord by Seven:

$\frac{7}{1}$ – Seventh Harmonic
Double Octave plus
Sub-minor Seventh

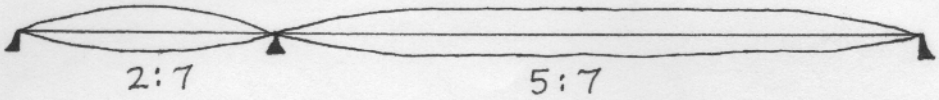
$\frac{7}{6}$ – Sub-minor Third
Sharp Second





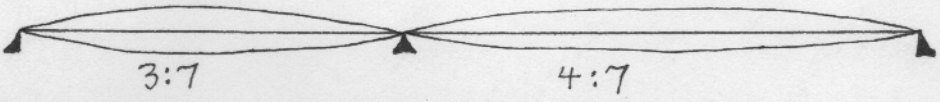
$\frac{7}{2}$ – Octave plus
Sub-harmonic Seventh

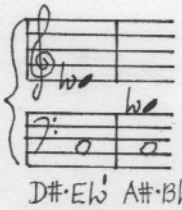
$\frac{7}{5}$ – Septimal Tritone

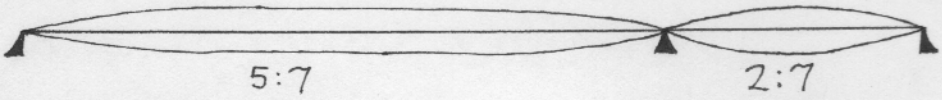


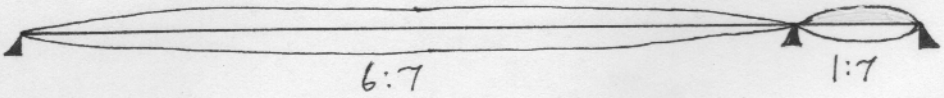
$\frac{7}{3}$ – Octave plus
Sub-minor Third
or sharpened second

$\frac{7}{4}$ – Sub-minor Seventh or
Sharpenned Sixth



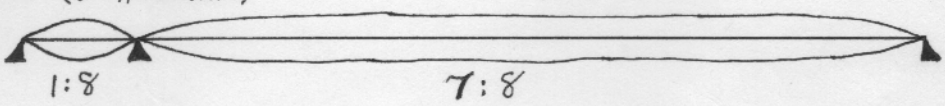


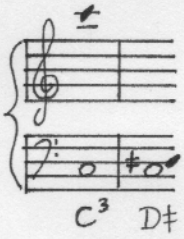




$\frac{8}{1}$ – Triple
Octave
(8th Harmonic)

$\frac{8}{7}$ – Septimal Wholetone
Semi-sharp Second (Super-second)





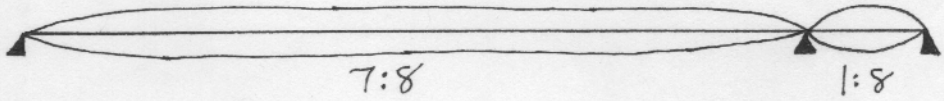
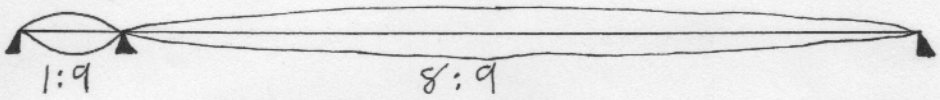


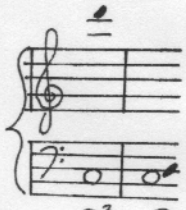
Figure 17: Division of the Monochord by Nine :

$\frac{9}{1}$ - 9th Harmonic. Triple Octave plus Major Wholetone

$\frac{9}{8}$ - Major Wholetone Pythagorean Wholetone



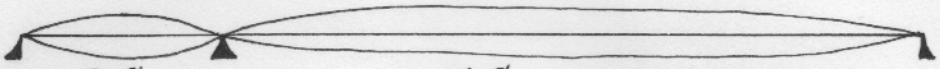
1:9 8:9



D^3 D

$\frac{9}{2}$ - Double Octave plus Major Wholetone

$\frac{9}{7}$ - Flattened Fourth Septimal Major Third



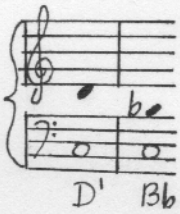
2:9 7:9

D^2 E♯·Fb

$\frac{9}{4}$ - Major Ninth

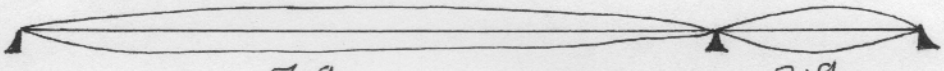
$\frac{9}{5}$ - Just Minor Seventh

4:9 5:9

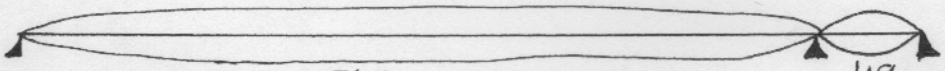


D^1 Bb

5:9 4:9



7:9 2:9

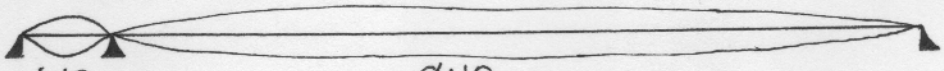


8:9 1:9

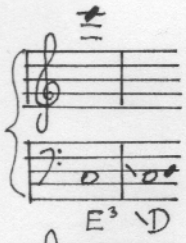
Figure 18: Division of the Monochord by 10 :

$\frac{10}{1}$ - 10th Harmonic. Triple Octave plus Major Third.

$\frac{10}{9}$ - Minor (Just) Wholetone



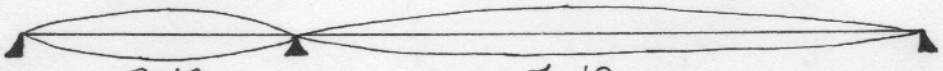
1:10 9:10



E^3 ♭D

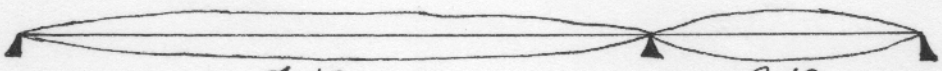
$\frac{10}{3}$ - Just Major Thirteenth

$\frac{10}{7}$ - Just Septimal Flattened Fifth

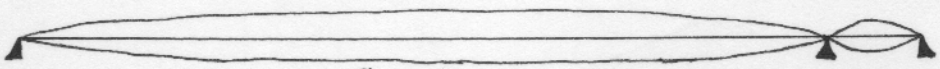


3:10 7:10

A^1 Gb

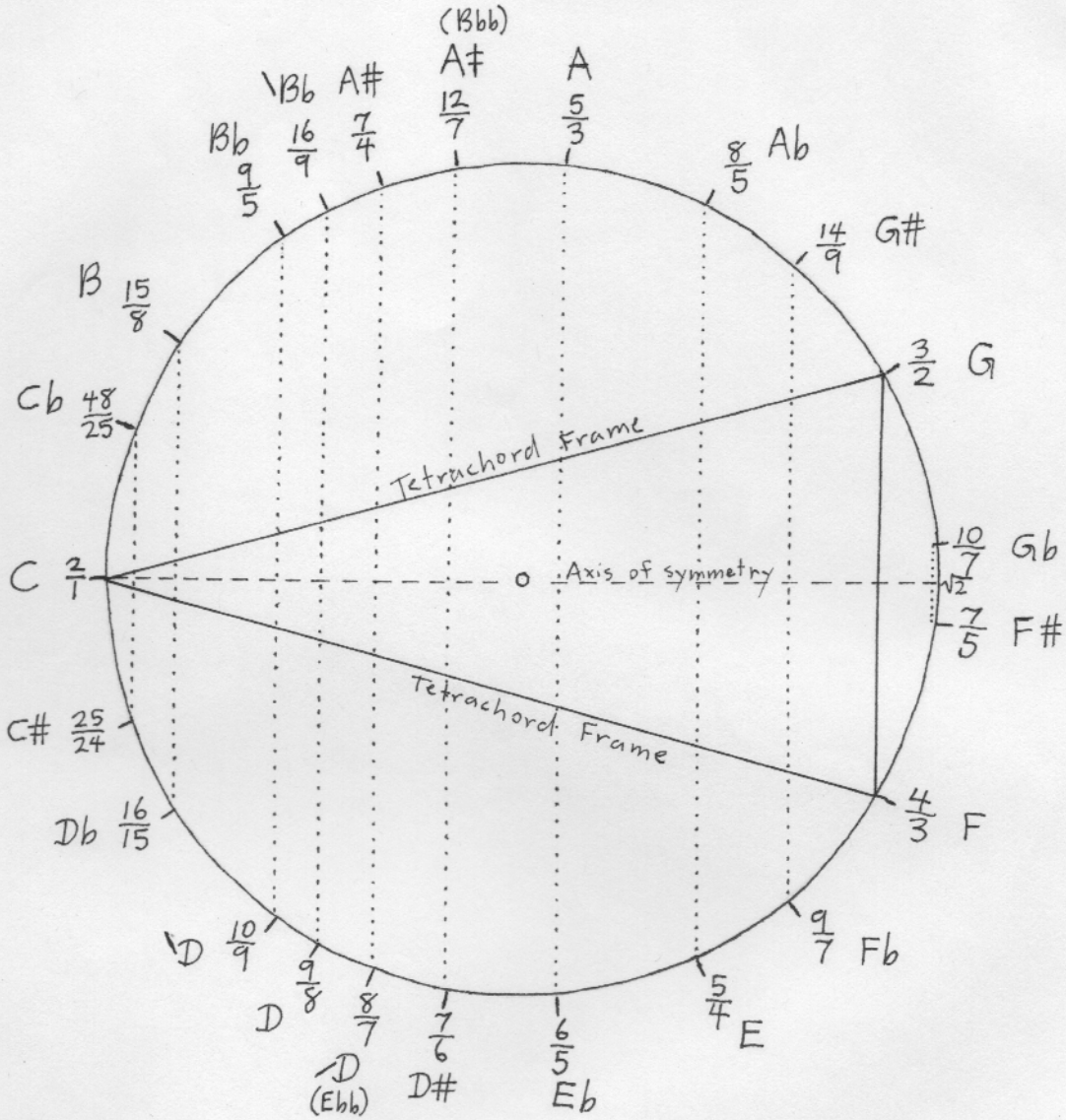


7:10 3:10



9:10 1:10

Figure 19: Prominant Just Intonation Ratios calibrated on the Circular Graph in order to view their metrical properties:



These ratios also appear in the following pattern on the Just lattice:

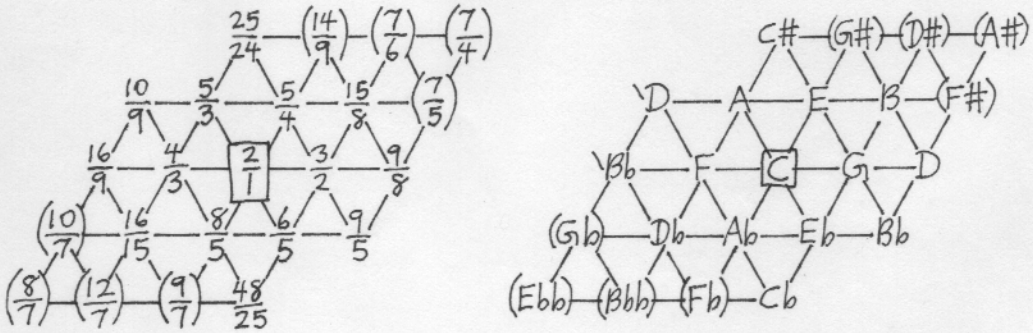
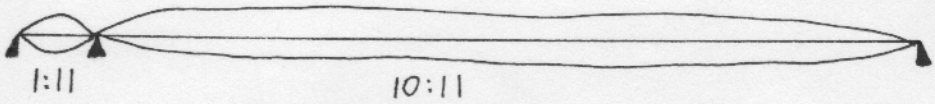


Figure 20: The division of the Monochord into eleven:

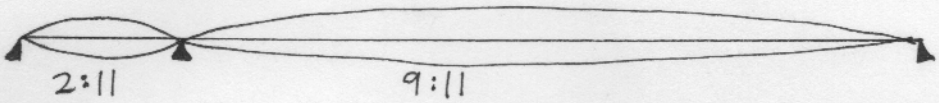
$\frac{11}{1}$ - 11th Harmonic Triple Octave plus Semi-sharp Fourth

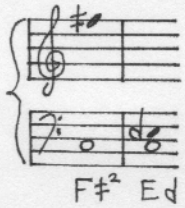
$\frac{11}{10}$ - Neutral Second ("3/4ths tone")



$\frac{11}{2}$ - Double Octave plus Semi-sharp Fourth

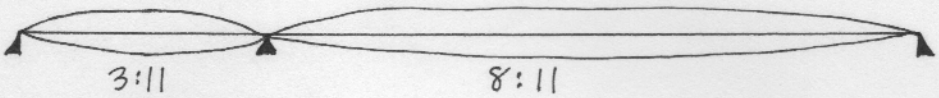
$\frac{11}{9}$ - Neutral Third

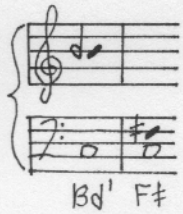




$\frac{11}{3}$ - Octave plus a semi-flat Seventh

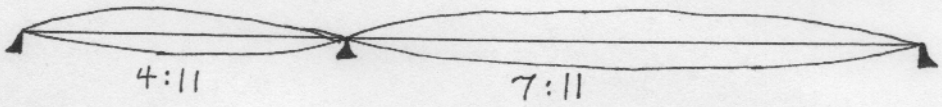
$\frac{11}{8}$ - Semi-sharp Fourth

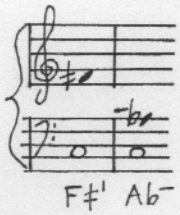




$\frac{11}{4}$ - Octave plus Semi-sharp Fourth

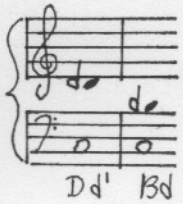
$\frac{11}{7}$ - 11-Limit (Dark) Minor Sixth (Close to Pythagorean Minor Sixth)

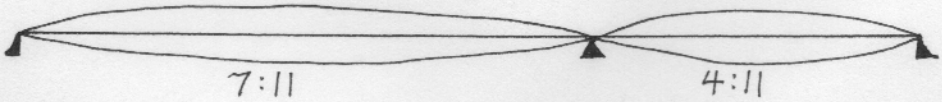


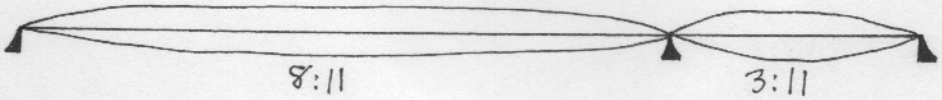


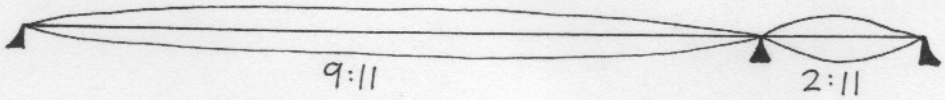
$\frac{11}{5}$ - Octave plus a neutral Second

$\frac{11}{6}$ - Neutral Seventh









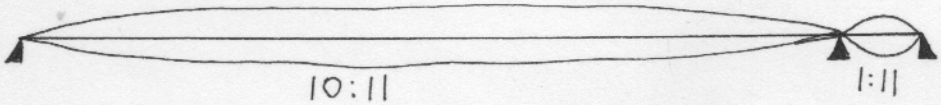
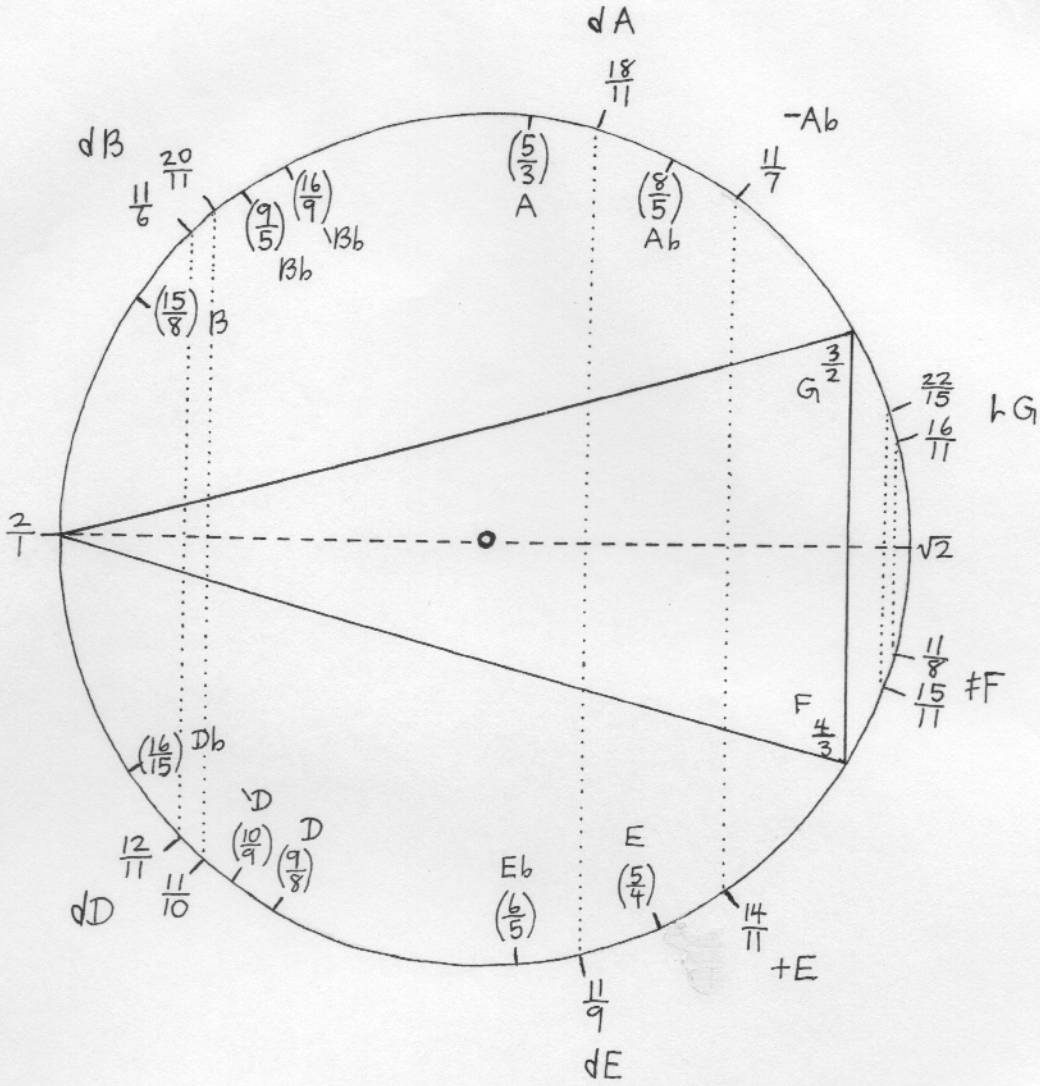


Figure 21 : Prominant 11-Limit Ratios drawn on the Circular Graph showing their metrical Properties :



Intervals from the 11 Division and their inversions :

11/10 – Neutral Second (3/4ths tone) – 20/11 – Neutral Seventh

11/9 – Neutral Third — 18/11 – Neutral Sixth

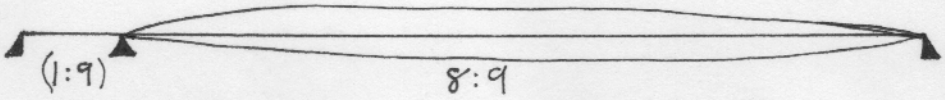
11/8 – Semi-sharp Fourth — 16/11 – Semi-flat Fifth

11/7 – "Dark" Minor Sixth — 14/11 – "Bright" Major Third

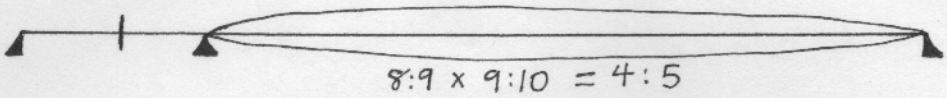
11/6 – Neutral Seventh — 12/11 – Neutral Second

Figure 22: Problem- to find the sum of a Major Wholetone and a Minor Wholetone (ie 9/8 x 10/9):

Solution: First, measure off a Major Wholetone (8/9):



Second, use the remaining length and measure off a Minor Wholetone (9/10):



Now this computation will result in the movable bridge being positioned at the Major Third (4/5). Errors depend on the accuracy of the 8:9 and 9:10 measurements.

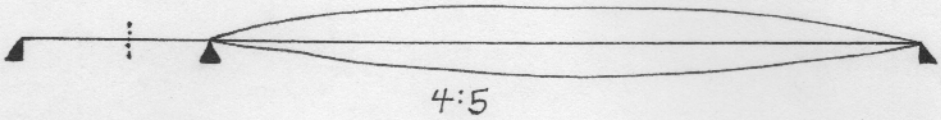
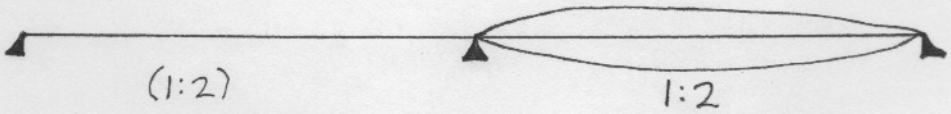
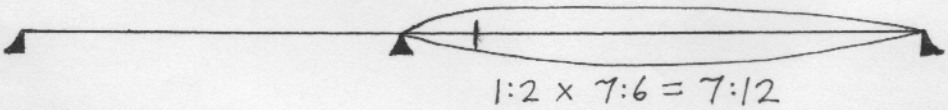


Figure 23: Problem – to subtract a sub-minor Third (frequency ratio 6/7) from the octave (2/1), ie. 2/1 x 6/7:

Solution: First, measure off an octave (1/2):



Second, use the remaining length and subtract a subminor third. Remember the inversion – we must multiply the length by 7/6:



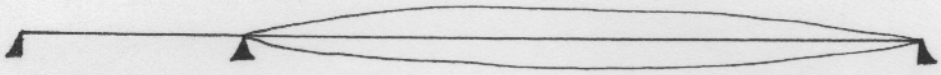
Now this computation results in the bridge placement being 7/12 – the Super-Major (Septimal) Sixth. Thus the string may be lengthened as well as shortened, as long as it is not lengthened beyond the full length. Greek monochord tuning tradition dictated beginning with the octave and progressively lengthening the string in tuning scales. The Greeks conceived a scale as a falling pattern, not a rising one as did the mediaeval theorists.

Figure 24: Problem: compute the Pythagorean Semitone or Limma (Frequency ratio 256/243) on the monochord:

Solution: When dealing with a complex ratio, if one does not wish to measure it directly on the instrument, the best approach is to break the fraction down into simple componants. In this case, string ratio $243/256 = 3/4 \times 9/8 \times 9/8$. If these factors are not obvious, the best approach is to find the prime factors of the complex ratio. Thus $243 = 3^5$ and $256 = 2^8$. Then these prime factors may be combined to form simple ratios as above.

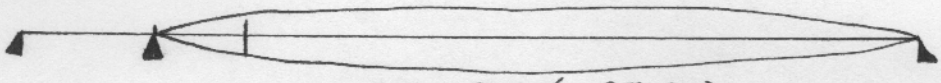
In applying $243/256 = 3/4 \times 9/8 \times 9/8$ to the instrument:

First, measure off the 3/4 ratio



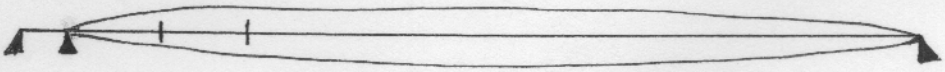
(1:4) 3:4

Second, use the remaining length and measure off a Major Wholetone down in pitch (string length 9/8):



3:4 x 9:8 (= 27:32)

Third, use this remaining length to measure off another Major Wholetone down in pitch:



3:4 x 9:8 x 9:8 = 243/256

This position will now give us the desired ratio 243/256.

Obviously, since there are three measurements here, there is greater likelihood of innaccuracy — since errors are accumulative. The key to reducing the amount of error is to keep the number of steps to a minimum. For instance, the string ratio 243/256 could also have been seen as a Pythagorean Minor Third (27/32) subtract a Wholetone (9/8) so that only two measurements need be made instead of three. Thus the efficient use of the monochord entailed some skill in handling the arithmetic of ratios.

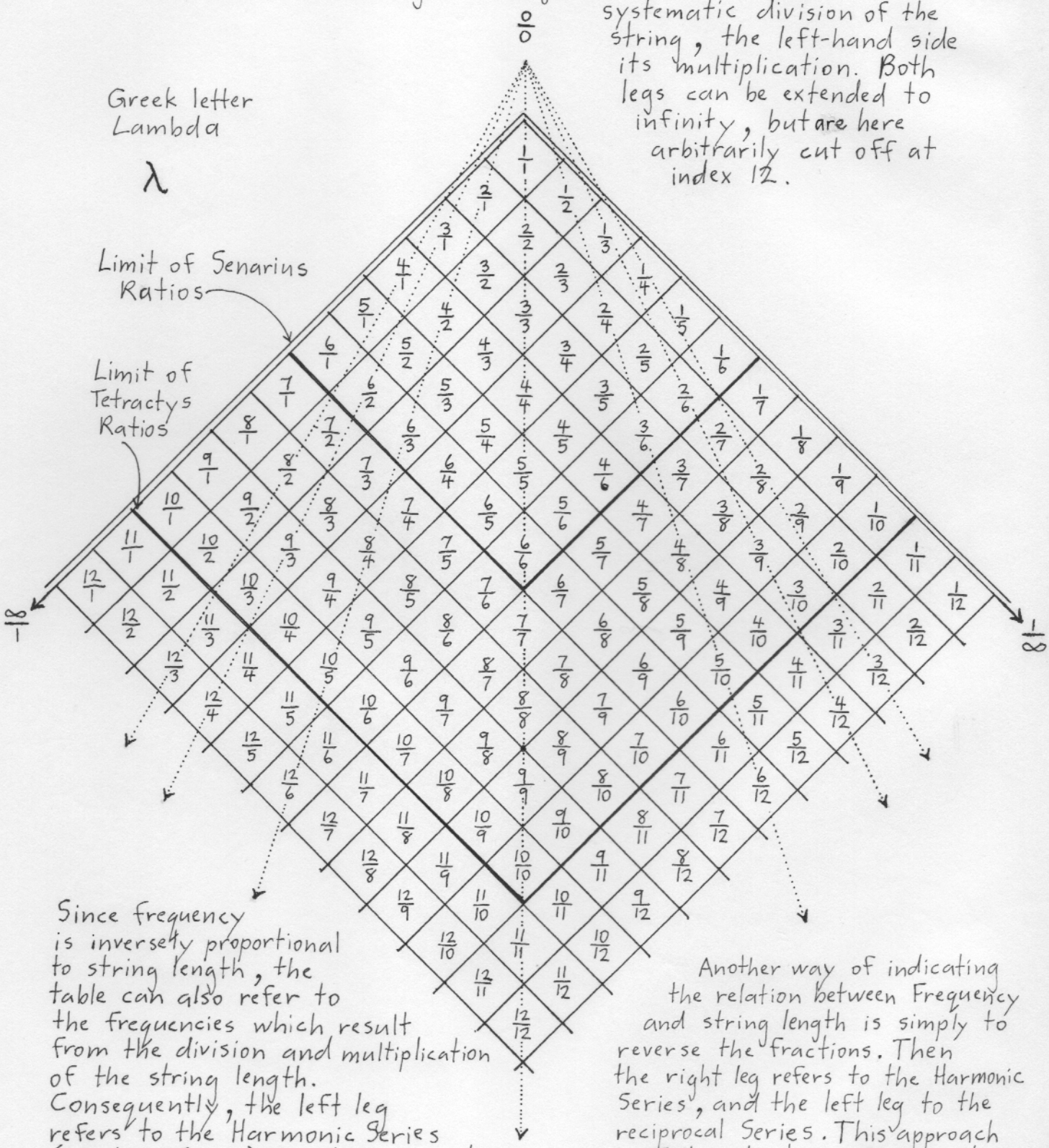
FC

Figure 25: The Lambdoma (Pythagorean Table):

Showing the reciprocity between the division and the multiplication of the monochord string. The right-hand side follows the systematic division of the string, the left-hand side its multiplication. Both legs can be extended to infinity, but are here arbitrarily cut off at index 12.

Greek letter Lambda

λ

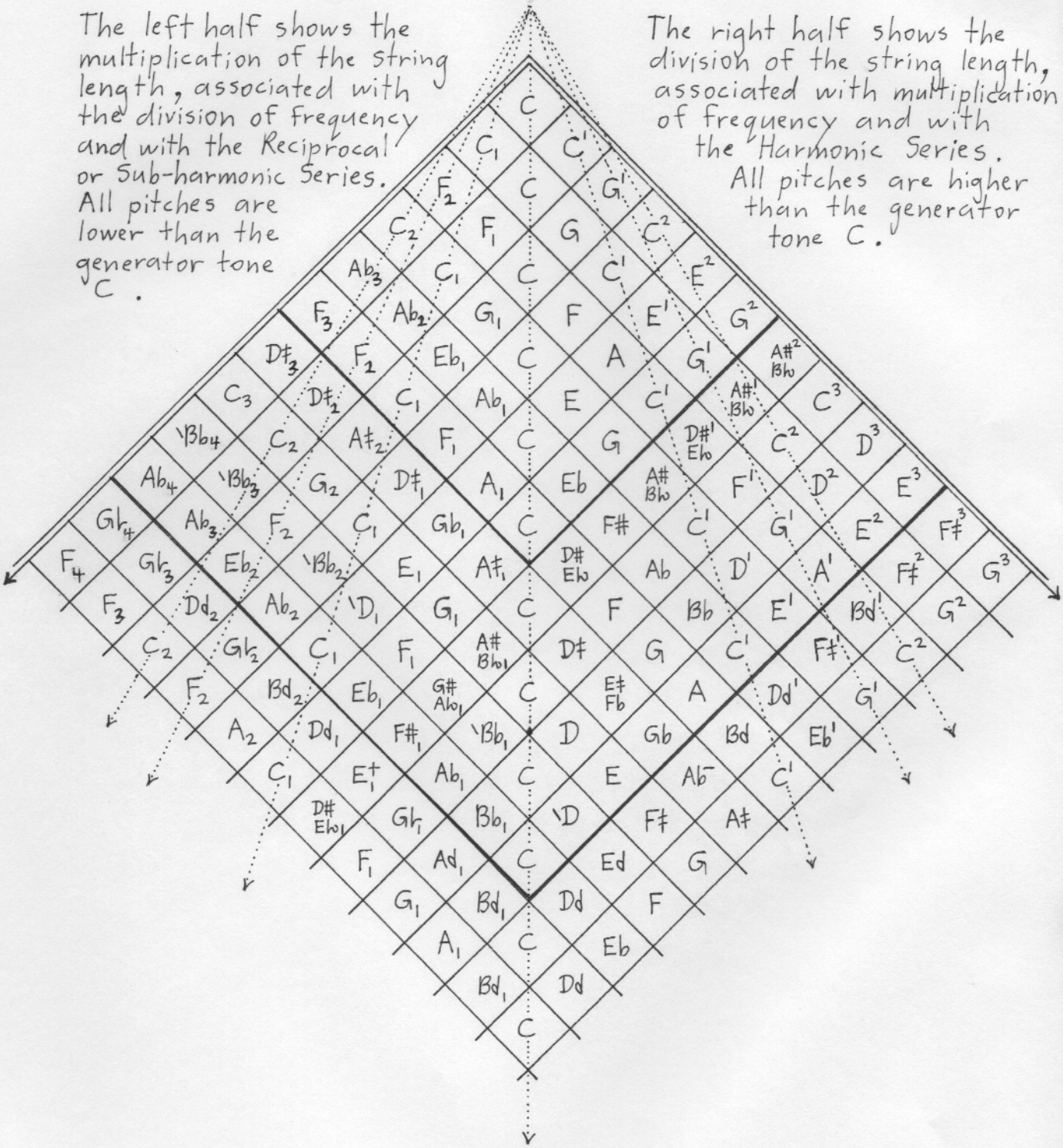


Since frequency is inversely proportional to string length, the table can also refer to the frequencies which result from the division and multiplication of the string length. Consequently, the left leg refers to the Harmonic Series (Overtone Series), and the right leg refers to the Sub-harmonic Series (reciprocal overtone series).

Another way of indicating the relation between Frequency and string length is simply to reverse the fractions. Then the right leg refers to the Harmonic Series, and the left leg to the reciprocal Series. This approach is followed when referring to the following diagram which converts ratios to musical pitches.

Figure 26: The Lambdoma (Pythagorean Table):

With ratios converted to pitch names. Superscripts refer to the octave number above the generator tone, and subscripts refer to the octave number below the generator tone.

The left half shows the multiplication of the string length, associated with the division of frequency and with the Reciprocal or Sub-harmonic Series. All pitches are lower than the generator tone C.

The right half shows the division of the string length, associated with multiplication of frequency and with the Harmonic Series. All pitches are higher than the generator tone C.